TORONTO METROPOLITAN UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE

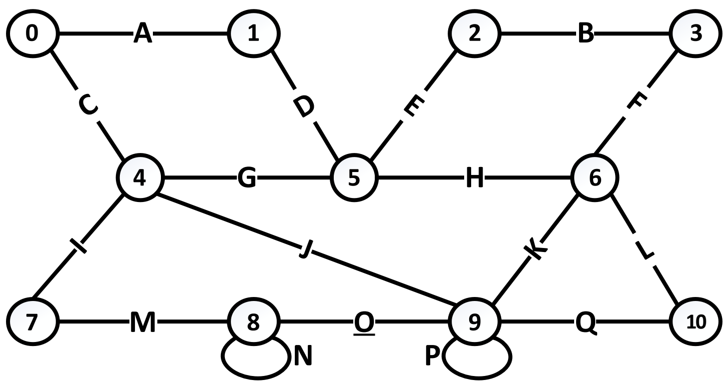
CPS 420
MIDTERM
WINTER 2024

INSTRUCTIONS

- This exam is 120 minutes long.
- This exam is out of 50 and is worth 25% of the course mark.
- This is a closed book exam. However, one double-sided letter-sized crib sheet is allowed.
- This exam is double-sided and has 10 pages including this front page. The last 2 pages are blank. Therefore there are 7 pages of questions: pages 2 to 8 inclusive.
- Please answer all questions directly on this exam. If you need extra space to finish answering questions, please do so on pages 9 and 10 and indicate very clearly on the original page of each question on which page the rest of your answer can be found.

PART A – GRAPH THEORY – 25 MARKS

A1 Euler and Hamiltonian Circuits (6 marks)



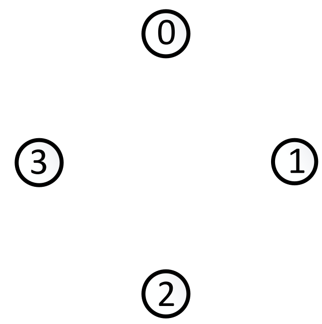
a) Give a **Euler** circuit starting at vertex 0 for the graph above, **or** explain why this graph does not have a **Euler** circuit. If you are giving a circuit, just list the edges in order.

b) Give a **Hamiltonian** circuit starting at vertex 0 for the graph above, **or** explain why this graph does not have a **Hamiltonian** circuit. If you are giving a circuit, just list the edges in order.

A2 Matrices in Graph Theory (5 marks)

a) On the diagram on the right, draw the **directed** graph G described by the adjacency matrix on the left (i.e. add the edges in G)

	0	1	2	3
0	1	1	0	1
1	1	0	0	0
2	1	1	1	0
3	0	0	0	0



b) Fill out the matrix A below which is defined as follows:

A(i,j) = number of walks of length 2 from vertex i to vertex j in the graph G.

i \ j	0	1	2	3
0				
1				
2				
3				

A3 Simple Graphs (14 marks)

a) Defining the following simple connected graphs for $n \geq 3$

- The *complete graph K_n* is the graph with n vertices and an edge between every two distinct vertices
- The *cycle graph C_n* is the graph that consists of a single cycle with n vertices
- The *star graph S_n* is the tree with 1 internal node and n leaves
- The *wheel graph W_n* is the graph formed by adding a vertex to C_n and an edge between that new vertex and each of the vertices of C_n

Draw the graphs (vertices and edges do not need to be labelled):

K_3	C_3	S_3	W_3
K_4	C_4	S_4	W_4
K_5	C_5	S_5	W_5

b) What is the relationship between C_n, S_n, and W_n?

c) For a simple connected graph G with set of vertices V(G) and set of edges E(G), define:

- The *distance d(v,w)* between two vertices v and w is the length (number of edges) of the **shortest walk** in G connecting v and w.
- The *eccentricity e(v)* of a vertex v is the **longest distance** between v and another vertex of G:
  $e(v) = \max\{ d(v,w) \mid w \in V(G) \}$
- A *center of G* is a vertex v of G with **minimal eccentricity**.
- The *radius ρ(G)* of G is the eccentricity of one of G's centers:
  $\rho(G) = \min \{ e(v) \mid v \in V(G) \}$
- The *diameter δ(G)* is the **longest distance** between two vertices of G:
  $\delta(G) = \max\{ d(v,w) \mid v,w \in V(G) \} = \max \{ e(v) \mid v \in V(G) \}$

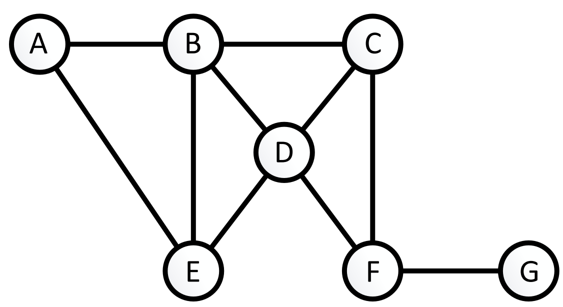
For the graph G underneath:

i. Circle all the centers of G on the diagram

ii. Radius $\rho(G) =$

iii. Diameter $\delta(G) =$

iv. Give one of the walks (i.e. list the vertices of the walk in order) that is the length of this diameter:



d) For the special graphs defined in a) when $n \geq 3$, give the radius and diameter as functions of n.

Graph G	K_n	C_n	S_n	W_n
Radius ρ(G)				
Diameter δ(G)				

PART B – SEQUENCES, RECURRENCE RELATIONS – 10 MARKS

Given the sequence a_n defined with the recurrence relation:

$a_0 = 3$
$a_k = 5a_{k-1} + 2k$ for $k \geq 1$

B1 Terms of the Sequence (5 marks)

Calculate a_1, a_2, a_3, a_4, a_5
Keep your intermediate answers as you will need them in the next question.

B2 Iteration (5 marks)

Guess a concise formula for a_n that uses the sum (Σ) and/or product ($\prod$) notation.

PART C – INDUCTION – 15 MARKS

Given the sequence b_n defined recursively as:
$b_0 = 3$, $b_1 = 7$
$b_k = 3b_{k-1} - 2b_{k-2}$ for $k \geq 2$

You will now prove by **strong induction** that a solution to this sequence is $b_n = 2^{n+2} - 1$.
No other method is acceptable.
Be sure to lay out your proof clearly and correctly and to justify every step.

C1 Problem Statement (2 marks)

The conjecture that you are proving, is expressed symbolically in the form $\forall n \in D, P(n)$.

- (1 mark) What is the set D?

- (1 mark) What is the predicate function P(n)?

C2 Base Cases (4 marks) Prove your base cases here:

C3 Inductive step setup (3.5 marks)

- (2 marks) State the assumption in the inductive step and identify the inductive hypothesis.

- (1.5 marks) State what you will be proving in the inductive step.

C4 Remainder of Inductive step (5.5 marks).

Finish your proof here. Be sure to justify every step, particularly why the recursive definition can be applied and where the inductive hypothesis is applied.

THIS PAGE IS INTENTIONALLY LEFT BLANK AND CAN BE USED FOR ROUGH WORK OR TO CONTINUE ANSWERING AN EARLIER QUESTION.

WORK ON THIS PAGE WILL ONLY BE GRADED IF SPECIFICALLY REQUESTED ON ONE OF PAGES 2 TO 7.

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